Homework 1
Solutions

1. (a) Let $x \in A - B$
Then $x \in A$ and $x \notin B$ ( the definition of $A - B$ )
So $x \in A$ and $x \in B^c$
So $x \in A \cap B^c$
Thus if $x \in A - B$, $x \in A \cap B^c$. This means $A - B \subseteq A \cap B^c$

Now suppose that $x \in A \cap B^c$
Then $x \in A$ and $x \in B^c$.
So $x \in A$ and $x \notin B$
So $x \in A - B$
Thus we can say that $A - B \subseteq A \cap B^c$
Since $A - B \subseteq A \cap B^c$ and $A \cap B^c \subseteq A - B$ we can conclude that $A - B = A \cap B^c$.

b) Let $x \in (A - B) \cup (A - B^c)$
then $x \in A - B$ or $x \in A - B^c$
if $x \in A - B$ then $x \in A$ and $x \notin B$. But this would mean $x \in A$
if $x \in A - B^c$ then $x \in A$ and $x \notin B^c$. But this also means $x \in A$.

so either way, $x \in A$.
Thus, $(A - B) \cup (A - B^c) \subseteq A$.

Now suppose $x \in A$.
if $x \in B$ then $x \in A \cap B = A - B^c$
so $x \in (A - B) \cup (A - B^c)$
if $x \notin B$, then $x \in A - B$ and thus $x \in (A \cap B) \cup (A \cap B^c)$
in both cases, $x \in (A - B) \cup (A - B^c)$.
Thus $A \subseteq (A - B) \cup (A - B^c)$ and since $(A - B) \cup (A - B^c) \subseteq A$ then $A = (A - B) \cup (A - B^c)$

c) Our goal is to show that if $a \in A$ then $a \in B$ <u>and</u> that if $b \in B$ then $b \in A$
We can do this in one step
Let $a \in A$ and $b \in B$.
from the definition, this means $(a, b) \in A \times B$
Since we know that $A \times B = B \times A$ in this case then $(a, b) \in B \times A$
but this means that $a \in B$ and $b \in A$.
so any $a \in A$ means $a \in B$ and any $b \in B$ means $b \in A$.
so $A \subseteq B$ and $B \subseteq A$
Thus $A = B$

d) Suppose $p \in A \times (B \cap C)$.
then there is some $a \in A$ and some $x \in B \cap C$ so that $p = (a, x)$
Since $x \in B$, then $p = (a, x) \in A \times B$
Since $x \in C$, then $p = (a, x) \in A \times C$
So $p \in A \times B$ and $p \in A \times C$.
thus, since for any $p \in A \times (B \cap C)$ is also an element of $(A \times B) \cap (A \times C)$
then $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

1c) continued

now suppose $p \in (A \times B) \cap (A \times C)$

Since $p \in A \times B$, there is some $a \in A$ and $x \in B$ s.t. $p = (a, x)$

Since $p \in A \times C$, there is some $a \in A$ and some $x \in C$ s.t. $p = (a, x)$

So $a \in A$ and $x \in C$ and $x \in B$ which implies $a \in A$ and $x \in B \cap C$

thus $p = (a, x) \in A \cap (B \cap C)$

So $p \in A \times (B \cap C)$

So $p \in (A \times B) \cap (A \times C)$ implies $p \in A \times (B \cap C)$ and $p \in A \times (B \cap C)$ implies $p \in (A \times B) \cap (A \times C)$

thus $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$ and $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

So $A \times (B \cap C) = (A \times B) \cap (A \times C)$

2.

(1) clearly $\emptyset \in \mathcal{F}$

(2) $\emptyset \in \mathcal{F}$ and $\emptyset^c = \Omega \in \mathcal{F}$

$A \in \mathcal{F}$ and $A^c \in \mathcal{F}$

$A^c \in \mathcal{F}$ and $(A^c)^c = A \in \mathcal{F}$

$\Omega \in \mathcal{F}$ and $(\Omega)^c = \emptyset \in \mathcal{F}$

c) Let $A_1, A_2, A_3, \ldots$ be a countable number of sets with each $A_i \in \mathcal{F}$.

<u>case I</u>: All $A_i = \emptyset$

then $A_1 \cup A_2 \cup \ldots = \emptyset$ and $\emptyset \in \mathcal{F}$ ✓

<u>case II</u>: All $A_i$ either $A$ or $\emptyset$

then $A_1 \cup A_2 \cup A_3 \cup \ldots = A \cup \emptyset = A \in \mathcal{F}$

<u>case III</u>: All $A_i$ either $\emptyset$ or $A^c$

then $A_1 \cup A_2 \cup \ldots = A^c \cup \emptyset = A^c \in \mathcal{F}$

<u>Case IV</u>: At least one $A_i$ is $A$, at least one is $A_i^c$, and all others are $\emptyset$

then $A_1 \cup A_2 \cup \ldots = A \cup A^c \cup \emptyset = \Omega \in \mathcal{F}$

<u>Case V</u>: At least one $A_i = \Omega$

Then $A_1 \cup A_2 \cup \ldots = A_1 \cup A_2 \cup \ldots \cup A_k \cup \Omega \cup A_{k+2} \cup \ldots = \Omega \in \mathcal{F}$

there are no other cases, and thus the third property is satisfied

Thus $\mathcal{F}$ satisfies the three properties and we can define valid probabilities on this set

Note: We call such sets "sigma algebras"

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$F([a,b]) = \int_a^b 3e^{-3x}dx = -e^{-3x}\Big|_a^b = (-e^{-3b}) - (-e^{-3a}) = e^{-3a} - e^{-3b}$

Notice: $\frac{d}{dx}(e^{-3x}) = -3e^{-3x}$ which is always negative.
this means that $e^{-3x}$ is always decreasing and thus the maximum occurs when $x=0$ and as $x \to \infty$ the value gets smaller and smaller.

Since $e^{-3x}$ is decreasing, the largest value of $e^{-3a} - e^{-3b}$ will come when $a=0$ and $b \to \infty$

additionally since $\lim_{x\to\infty} e^{-3x} = 0$, $e^{-3a} - e^{-3b} \le 1 - 0 = 1$ for all $a, b$

so it is bounded above by 1

also since $e^{-3a} \ge e^{-3b}$ for all $b \ge a$, $e^{-3a} - e^{-3b} \ge 0$ for all $b \ge a$

so $0 \le e^{-3a} - e^{-3b} \le 1$

b) $A \subset B \Rightarrow A = [a_1, a_2]$, $B = [b_1, b_2] \Rightarrow b_1 < a_1 < a_2 < b_2$

$$F(B) = \int_{b_1}^{b_2} 3e^{-3x}dx = \int_{b_1}^{a_1} 3e^{-3x}dx + \int_{a_1}^{a_2} 3e^{-3x}dx + \int_{a_2}^{b_2} 3e^{-3x}dx$$

$$= F([b_1, a_1]) + F([a_1, a_2]) + F([a_2, b_2])$$

$$\ge 0 + F(A) + 0$$ (since $F$ is bounded below by 0)

$$= F(A)$$

So $F(B) \ge F(A)$

c) generally, $F([a,b]) = e^{-3a} - e^{-3b}$

for $0 \le a \le b \le n$

$F([0,n]) = e^{-3\cdot 0} - e^{-3n} = 1 - e^{-3n}$ and as $n \to \infty$ $F([0,n]) \to 1$

we could also write that as

$$F([0,n]) = \int_0^a 3e^{-3x}dx + \int_a^b 3e^{-3x}dx + \int_b^n 3e^{-3x}dx$$

$$= F([0,a]) + F([a,b]) + F([b,n])$$

so $\lim_{n\to\infty} F([0,n]) = \lim_{n\to\infty} \left( F([0,a]) + F([a,b]) + F([b,n]) \right)$

$= F([0,a]) + F([a,b]) + \lim_{n\to\infty} F([b,n])$

or

$1 = F([0,a]) + F([a,b]) + \lim_{n\to\infty} F([b,n])$

which gives

$F([a,b]) = 1 - F([0,a]) - \lim_{n\to\infty} F([b,n])$

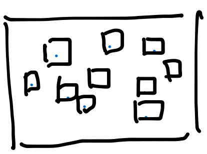
4. a) There is no snow outside, so no matter what I measure the volume of snow above will be 0.

b) Volume = length · width · height

$= (10m) \cdot (10m) \cdot (1m)$

$= 100\ m^3$

c)



with 10 missing points, we can say that $100\ m^3$ would be an overmeasurement.

If we measure the field as normal, but exclude small rectangles around the missing points, say squares with length $1/n$ then we would be under measuring the true volume.

If we let our true volume be $V$, $V_*$ = overmeasurement and $V^*$ be our under measurement, then

$V^* = 100 - (\#\text{missing points}) \cdot \frac{1}{n^2}$

$V_* = 100$ and

and since $V^* \leq V \leq V_*$ then

$100 - (\#\text{points}) \frac{1}{n^2} \leq V \leq 100$

as $n \to \infty$, we get $V_* \to 100$

that means that as $n \to \infty$ $V^* \to 100 \leq V \leq 100$

so, our under and over measurements both converge to $100\ m^3$

and since $V$ is "pinched" between them, $V = 100\ m^3$ must be our true volume.

d) 10 points 1000 points n points - it won't matter. the above approach will work